AMERICAN UNIVERSITY OF BEIRUT
Mathematics Department-FAS

MATH 251
FINAL EXAM
SPRING 2010-2011
Closed Book, 2hours

STUDENT NAME	
ID NUMBER	

Problem	Out of	Grade
1	15	
2	20	
3	20	
4	15	
5	15	
6	15	
TOTAL	100	

1. **(15 points)** For which values of x is there a difficulty in computing $f(x)$? What is the problem ? What remedy do you suggest ?

   - (7 points) $f(x) = x + \sqrt{x^2 + 4}$

   - (8 points) $f(x) = \tan(x) - 1$
     (N.B. Determine only the smallest positive value of x that causes a problem.)

2. **(20 points)** Consider the function $f(x) = x^2 - a$ where $1 < a \leq 2$.

   (a) (6 points)Write Newton's formula: $r_n = g(r_{n-1})$ that computes the positive root of the function f. Is there any restriction on the initial condition r_0? Justify your answer.

   (b) (2 points)Find the minimum number of flops necessary in computing one iteration $r_n = g(r_{n-1})$.

(c) (6 points)Prove that:

$$(r_n - r) = \frac{1}{2r_{n-1}}(r_{n-1} - r)^2$$

then show that for $n \geq 1$:

$$|r_n - r| = \frac{1}{2}(r_{n-1} - r)^2 < \frac{1}{2^n}(r_0 - r)^{2^n}$$

(d) (6 points)Let $r_0 = 3/2$. Find an estimate on the least value of n for which:

$$|r_n - r| < 2^{-50}$$

<u>Hint:</u> Using the result in (c), show first that $|r_n - r| < 2^{(-2^n)}$

3. **(20 points)** Consider the following set of data:

$$D_n = \{(x_i, y_i) | i = 0, ..., n \text{ where } y_i = f(x_i) \text{ , and } x_{i+1} - x_i = h,\ 0 < h \leq 1\}$$

(a) (6 points) Write first the **Central difference** formula $\psi_h(f(x_i))$ that approximates the first derivative $f'(x_i)$, then derive the expression of the error series $\epsilon(h) = f^{'}(x_i) - \psi_h(f(x_i))$ in the form:

$$\epsilon(h) = c_1 h^{\alpha_1} + c_2 h^{\alpha_2} + c_3 h^{\alpha_3} + ...,$$

by determining the values of the constants $\{\alpha_1,\ \alpha_2,\ \alpha_3, ...\}$.

- $\psi_h(f(x_i)) =$

- $\epsilon(h) =$

(b) (6 points) Based on the Central difference formula, derive Richardson extrapolation operators of orders 1 and 2 and the order of their error series.

- 1st order Richardson extrapolation operator:

  $\psi_h^1(f(x_i)) =$

  Corresponding Error = O(...........)

- 2nd order Richardson extrapolation operator:

  $\psi_h^2(f(x_i)) =$

  Corresponding Error = O(...........)

(c) (8 points) Let $f(x) = 3xe^x$. For the purpose of approximating $f'(2.0)$ and then improving this approximation, fill in the empty slots in the following table adequately, starting with $h_0 = 0.5$. Express all the results obtained in $F(10, 5, -15, +15)$.

h	$\psi_h(.)$	$\psi_h^1(.)$	$\psi_h^2(.)$
$h_0 = 0.5$			
$h_0/2$			
$h_0/4$			

Best approximation to $f'(2.0)$:

4. **(15 points)** Consider the following set of data, where the partition points $\{x_i\}_{i=0}^n$ are equally spaced:

$D_n = \{(x_i, y_i)|i = 0, ..., n\,;\ a = x_0, b = x_n;\ h = x_{i+1} - x_i;,\ y_i = f(x_i)\,;\}$

(a) (1pt)Based on D_n, approximate the integral $I = \int_a^b f(x)dx$ using the Composite Trapezoidal rule:

$I \approx T(h) =$

(b) (4 points) Complete the following MATLAB function that approximates the integral $I = \int_a^b f(x)dx$ using the Composite Trapezoidal rule. This function takes as inputs:

- a function f
- 2 real numbers $a < b$
- a positive number h, with $0 < h \leq 1$ where $n = \frac{b-a}{h}$ is the number of subintervals determined by the partition points.

```
function I = CompositeTrapezoid(f, a, b, h)
```

(c) (2 points)Assume $f \in C^\infty[a, b]$. Write the Romberg extrapolation formulae of orders 1,2 then k, based on the Composite Trapezoidal rule (Do not derive the formulae):

i. $R^1(h) =$

ii. $R^2(h) =$

iii. $R^k(h) =$

(d) (8 points)Complete the following MATLAB function that applies k iterations of the Romberg process based on the Composite Trapezoidal rule. The results are displayed in a lower triangular matrix R of size $(k+1) \times (k+2)$. (Call for the previous function if adequate)

```
% This algorithm applies the Romberg process  up to kth order
% Input : a function f, the limits of integration a and b and
% k: the number of iterations
% Output :  a lower triangular matrix R of size (k+1) x (k+2).

function R = RombergCompositeTrapezoid(f,a, b, k)
R = zeros(k+1,k+2) ;
% The first column of R consists of all values of h, starting with the
% largest value  h=(b-a) and ending at h=(b-a)/2^k

R(1,1)=b-a ;
```

```
% The elements of the second column of R are  of all values of T(h) :
%  approximations  of I  using the  Composite  Trapezoidal rule
```

```
% All  elements of the remaining columns of R from the 3rd one till
% the  last  one are  the  Romberg approximations of all orders
```

```
% Display the  best approximation to I
```

```
end
```

5. (15 points)Let $I = \int_0^2 3xe^x\, dx$.

- (12 points) How many partition points are needed if I is to be computed using the composite trapezoidal rule, up to two decimal places (rounding to the closest)? Compute I in that case.
  N.B The exact value of $I = 25.1671683$

.

- (3 points)In order to improve the approximation above, apply the Romberg process **once**, and write the formula of the 1st Romberg operator with the appropriate values of the new parameters. (Do not compute the new approximation to I).

6. **(15 points)** Consider the following Initial value Problem:

$$(IVP)\begin{cases} \frac{dy}{dt} = 2t + e^{-y}\,; \;\; t \in [1, 2.00] \\ y(1) = 3 \end{cases}$$

To solve (IVP) in $F(10, 4, -20, +20)$ (rounding to the closest):

(a) (3 points) (IVP) is first solved on $[1, 1.25]$ using the **discrete scheme** of Euler's method: (RK1).

- Write first the formulae for this scheme:

$$(RK1)\begin{cases} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ y_{i+1} = \dots\dots\dots\dots\dots\dots\dots\dots \end{cases}$$

- Use **1 step ONLY** of this scheme to approximate $y(1.25)$.

i	t_i	y_i	k_1	y_{i+1}
0	1.00	.	.	.

(b) (5 points)(IVP) is then solved on [1.25 , 1.50] using the **discrete scheme** of the 2nd order Runge Kutta method - Midpoint Rule: (RK2.M).

- Write then the formulae of this scheme:

$$(RK2.M)\begin{cases} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ y_{i+1} = \dots\dots\dots\dots\dots\dots\dots\dots \end{cases}$$

- Use **1 step ONLY** of this scheme to approximate $y(1.50)$.

i	t_i	y_i	k_1	k_2	y_{i+1}
0	1.25	.	.	.	.

(c) (6 points)Finally, (IVP) is then solved on [1.50 , 2.00] using the **discrete scheme** of the 2nd order Runge Kutta method - Trapezoid Rule (Heun's method): (RK2.T).

- Write then the formulae of this scheme:

$$(RK2.T)\begin{cases} \text{...} \\ \text{..} \\ y_{i+1} = \text{...} \end{cases}$$

- Use **2 steps ONLY** of this scheme to approximate $y(1.75)$ and $y(2.00)$.

i	t_i	y_i	k_1	k_2	y_{i+1}
0	1.50	.	.	.	.
1	.	.	.	.	.

(d) (1pt) The discrete solution of (IVP) is

$Y_4 = \{y_0 = 3, \text{...}\}$